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Abstract

A set $\mathcal{P}$ of nondegenerate convex polygons P in R^2, or polyhedra P in R^3, will be called *normal* if the intersection of any two of the P's of $\mathcal{P}$ is a face (in the case of polyhedra), an edge, a vertex, or empty. $\mathcal{P}$ is called *strongly normal* (SN) if it is normal and, for all P, $P_1, \ldots, P_n$, if each P_i intersects P and $I = P_1 \cap \ldots \cap P_n$ is nonempty, then I intersects P. The union of the $P_i \in \mathcal{P}$ that intersect $P \in \mathcal{P}$ is called the *neighborhood* of P in $\mathcal{P}$, and is denoted by $N_{\mathcal{P}}(P)$. We prove that $\mathcal{P}$ is SN iff for any $\mathcal{P}' \subseteq \mathcal{P}$ and $P \in \mathcal{P}'$, $N_{\mathcal{P}'}(P)$ is simply connected. Thus SN characterizes sets $\mathcal{P}$ of polyhedra (or polygons) in which the neighborhood of any polyhedron, relative to any subset $\mathcal{P}'$ of $\mathcal{P}$, is simply connected. Tessellations of R^2 or R^3 into convex polygons or polyhedra are normal, but they may not be SN; for example, the square and hexagonal regular tessellations of R^2 are SN, but the triangular regular tessellation is not.

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1 Introduction

In a recent report [1] on sets of tetrahedra, the authors introduced properties called normality and strong normality (SN), and showed that SN implies that the neighborhood of any tetrahedron (= the union of the tetrahedra in the set, including itself, that intersect it) is simply connected. In this note we generalize SN to sets of convex polyhedra (or polygons, in the plane), and show that it is in fact equivalent to simple-connectedness of neighborhoods. More precisely, we show that $\mathcal{P}$ is SN iff for any $\mathcal{P}' \subseteq \mathcal{P}$, the neighborhood $N_{\mathcal{P}'}(P)$ of any $P \in \mathcal{P}'$ is simply connected. Tessellations of the plane (or 3-space) into convex polygons (or polyhedra) are normal, but they may not be SN; for example, the square and hexagonal regular tessellations of the plane are SN, but the triangular regular tessellation is not.

Most of the research on digital geometry (connectedness, distance, thinning, convexity, etc.) has dealt with the square (or cubical) grid, though it is well known that other grids sometimes have computational advantages. A collection of references on digital geometry in which other grids, or other discrete spaces, are used can be found in [2]. These references deal primarily with other regular grids (hexagonal or triangular, in the plane); a notable exception was Sklansky's work on digital convexity [3], which was based on an arbitrary tessellation of the plane into convex polygons [4]. The present paper shows that from a topological standpoint, there may be significant differences between different types of tessellations, in both two and three dimensions.

2 The two-dimensional case

A set $\mathcal{P}$ of convex polygons in the plane will be called *normal* if

a) Each polygon is nondegenerate (i.e., has a nonempty interior)

b) The intersection of any two of the polygons is an edge, a vertex, or empty

Evidently, three or more polygons of $\mathcal{P}$ cannot share an edge; thus their intersection must be either a vertex or empty. If $\mathcal{P}$ covers the plane, it will be called a *tessellation.*

$\mathcal{P}$ is called *strongly normal* (SN) if it is normal and, for all $P, P_1, \ldots, P_n$ $(n \geq 1) \in \mathcal{P}$, if each P_i intersects P and $I = P_1 \cap \ldots \cap P_n$ is nonempty, then I intersects P. Note that if $n \geq 3$, I must be a single point and so must be a vertex of P. Note also that, like normality, strong normality is hereditary: If it holds for $\mathcal{P}$, it holds for any $\mathcal{P}' \subseteq \mathcal{P}$.

The *neighborhood* $N_{\mathcal{P}}(P)$ of P in $\mathcal{P}$ is the union of all $Q \in \mathcal{P}$ that intersect P (including P itself).

Theorem 1: If $\mathcal{P}$ is SN, then for any $\mathcal{P}' \subseteq \mathcal{P}$ the neighborhood $N_{\mathcal{P}'}(P)$ of any $P \in \mathcal{P}'$ is simply connected.

Proof: Any curve in $N_{\mathcal{P}'}(P)$ can be decomposed into nondegenerate arcs such that the interior of each arc is contained in at most one of the polygons, or the intersection of two of the polygons, of $N_{\mathcal{P}'}(P)$. Let C be such a curve that has a decomposition into as few such arcs as possible, say $C_1, \ldots, C_m$. If $m = 2$, C is contained in the union of two polygons of $N_{\mathcal{P}'}(P)$, and the intersection of these polygons is nonempty (it contains the common endpoints of the arcs); but the union of two intersecting convex polygons is evidently simply connected, so C can be deformed to a point, contradiction. For each i, let Q_i be (one of) the polygon(s) that contains C_i; by the minimality of m, successive Q_i's must be distinct. Let C leave Q_i and enter Q_{i+1} (modulo m) at p_i, which is a point of $Q_i \cap Q_{i+1}$. Since Q_i is convex, the arc C_i from p_{i-1} to p_i can be deformed into the line segment $p_{i-1}p_i$, which lies in Q_i. Suppose Q_{i-1}, Q_i, Q_{i+1} had a common point p. Then we could continuously deform C by moving p_{i-1} in $Q_{i-1} \cap Q_i$ and p_i in $Q_i \cap Q_{i+1}$ until they both coincide with p; this reduces $p_{i-1}p_i$ to the single point p, so that C_i is now a degenerate arc, contradicting the minimality of m. Hence any three successive Q's must be disjoint. Since $\mathcal{P}'$ is SN, $Q_{i-1} \cap Q_i$ and $Q_i \cap Q_{i+1}$ must both intersect P; hence we can continuously deform C by moving p_{i-1} in $Q_{i-1} \cap Q_i$ and p_i in $Q_i \cap Q_{i+1}$ until they both reach P. The line segment $p_{i-1}p_i$ then lies in P, so we can replace Q_i by P. As just shown, $Q_i = P, Q_{i+1}$, and Q_{i+2} must be disjoint; but this implies that $Q_{i+1} \cap Q_{i+2}$ must be disjoint from P, contradicting SN. □

Theorem 2: Let $\mathcal{P}$ be such that, for any $\mathcal{P}' \subseteq \mathcal{P}$ and any $P \in \mathcal{P}'$, $N_{\mathcal{P}'}(P)$ is simply connected; then $\mathcal{P}$ is SN.

Proof: We first show that if $R_1, \ldots, R_k$ is a minimal set of neighbors of P that violates SN, then there exist i, j such that $P \cap R_i$ and $P \cap R_j$ are disjoint. Note first that if $k = 2$, $P \cap R_1$ and $P \cap R_2$ must be disjoint (if not, $R_1 \cap R_2$ would intersect P and we would not have a violation of SN). For $k > 2$, if all the $P \cap R$'s are vertices of P they cannot all be the same (otherwise SN would not be violated); hence two of them are disjoint. If $P \cap R_i$ is an edge and some $P \cap R_j$ is a vertex, if it were a vertex of that edge we could eliminate R_j and still have a violation of SN, contradicting minimality; hence it is not a vertex of that edge, so is disjoint from $P \cap R_i$. Finally, suppose all the $P \cap R$'s are edges. An edge can intersect at most two other edges, so if there are three or more other edges, we have a disjoint pair. If there are only two other edges, they are disjoint unless P is a triangle; but in that case the intersection of all three R_i's is empty, so SN is not violated. Thus in all cases there exist R_i, R_j such that $P \cap R_i$ and $P \cap R_j$ are disjoint; and this implies that $R_i \cap R_j$ (which is nonempty, since it contains the intersection of all the R's) cannot intersect P, so that $\mathcal{P}' = \{P, R_i, R_j\}$ is in fact the minimal violation of SN.

Let p, p_i, p_j be points in $R_i \cap R_j$, $P \cap R_i$, and $P \cap R_j$, respectively. Thus p, p_i and p_j form a triangle T such that each of P, R_i and R_j contains an edge of T and does not contain T's third vertex. This also implies that T is nondegenerate. [Indeed, if its vertices were collinear, one of the edges of T would contain the other two and so one of P, R_i, and R_j would contain all three vertices, contradiction.] This also implies that no one of P, R_i and R_j can contain T. Now the interior of T is surrounded by the edges of T; hence it is surrounded by $P \cup R_i \cup R_j (= N_{\mathcal{P}'}(P))$. If we can show that the interior of T is not contained in $P \cup R_i \cup R_j$, it will follow that $N_{\mathcal{P}'}(P)$ has a hole, and hence is not simply connected.

Since no one of P, R_i, R_j can contain T, we are done unless at least two of them intersect the interior of T. We shall show that the interior contains a vertex. Suppose $R_i \cap R_j$ intersects the interior. If $R_i \cap R_j$ is contained in the interior, the interior contains a vertex of $R_i \cap R_j$. [By normality, $R_i \cap R_j$ must contain a vertex.] If $R_i \cap R_j$ intersects the interior but the interior contains no vertex, then $R_i \cap R_j$ must be an edge and must intersect the boundary of T at two points. But $R_i \cap R_j \cap$ (the boundary of T) is the point p. [$R_i \cap R_j$ cannot intersect p_ip_j. Suppose it intersects pp_i or pp_j (say the latter) at some point other than p.

Then by normality pp_j must be a subset of the edge $R_i \cap R_j$; but this implies that R_i contains p_j, contradiction.] Thus the interior contains a vertex, say v. To fill the 2D space around v at least three polygons must meet at v; but this implies that $P \cap R_i \cap R_j$ is nonempty, contradiction. □

The regular square or hexagonal tessellation of the plane is evidently SN; but the regular triangular tessellation is not. (For any triangle T, there are two triangles A, B that intersect T in vertices at opposite ends of an edge and that also share a vertex; thus $I = A \cap B$ is nonempty but does not intersect T.) Note that in the "subtessellation" obtained by omitting one of each pair of triangles that share an edge (e.g., omitting all the triangles whose bases face northward), the neighborhood of any triangle is in fact not simply connected (indeed, it has three holes).

3 The three-dimensional case

A set $\mathcal{P}$ of convex polyhedra in 3-space will be called *normal* if

a) Each polyhedron is nondegenerate (i.e., has a nonempty interior)

b) The intersection of any two of the polyhedra is a face, an edge, a vertex, or empty

Evidently, three or more polyhedra of $\mathcal{P}$ cannot share a face; thus their intersection must be either an edge, a vertex or empty. If $\mathcal{P}$ covers 3-space, it will be called a *tessellation.*

The neighborhood $N_{\mathcal{P}}(P)$, and strong normality, are defined as in the two-dimensional case.

Theorem 3: If $\mathcal{P}$ is SN, then for any $\mathcal{P}' \subseteq \mathcal{P}$, the neighborhood $N_{\mathcal{P}'}(P)$ of any $P \in \mathcal{P}'$ is simply connected.

Proof: Suppose $N_{\mathcal{P}'}(P)$ has a tunnel; then there exists a closed curve in $N_{\mathcal{P}'}(P)$ that cannot be reduced to a point. The proof that this contradicts SN is exactly as in the two-dimensional case, with "polygons" replaced throughout by "polyhedra", and "two" replaced by "two or more" in the first sentence.

Suppose next that $N_{\mathcal{P}'}(P)$ has a cavity K. P is the intersection of a finite number of half-spaces bounded by the planes containing its faces. Evidently, K cannot be contained in all of these half-spaces; thus there exists a plane Π containing a face of P, such that P is on one side of Π and some point of K is (strictly) on the other side. Since K is bounded, we can translate Π parallel to itself, away from P, until no point of K lies beyond it; let Π' be the position of Π when this happens, so that K intersects Π' but does not extend beyond Π'. Since K is bounded by a finite set of polyhedra belonging to $N_{\mathcal{P}'}(P)$, it has a polyhedral shape; thus it intersects Π' in a set of polygonal regions (possibly degenerate). Let p be a vertex of one of the regions. In a sufficiently small neighborhood of p, since p is a vertex, K cannot occupy the entire halfspace on the side of Π' toward Π; in fact, p must lie on at least three noncoplanar faces F_i of K. Let F_i lie in plane Π_i, and let Q_i be the polyhedron of $N_{\mathcal{P}'}(P)$ that bounds K along face F_i. Thus Π_i divides R^3 into the halfspaces H_i, H_i' such that each Q_i lies in H_i and, in a sufficiently small neighborhood of p, K lies in each H_i' and is their intersection. Thus this intersection lies on the side of Π' toward Π, so that the intersection of the H_i's, hence the intersection of the Q_i's, lies on the side of Π' away from Π, and this intersection is nonempty since it contains p. Since P lies on the side of Π away from Π', P is thus disjoint from the intersection of the Q_i's, contradicting SN. [A two-dimensional version of this proof could have been used to show that $N_{\mathcal{P}'}(P)$ (in 2D) cannot have a hole.] □

Theorem 4: Let $\mathcal{P}$ be such that, for any normal $\mathcal{P}' \subseteq \mathcal{P}$ and any $P \in \mathcal{P}'$, $N_{\mathcal{P}'}(P)$ is simply connected; then $\mathcal{P}$ is SN.

Proof: We first show that if $R_1, \ldots, R_k$ is a minimal set of neighbors of P that violates SN, then (a) there exist i, j such that $P \cap R_i$ and $P \cap R_j$ are disjoint, or (b) there exist i, j, k such that $P \cap R_i \cap R_j$, $P \cap R_j \cap R_k$ and $P \cap R_i \cap R_k$ are disjoint. Note first that if $k = 2$, $P \cap R_1$ and $P \cap R_2$ must be disjoint, since otherwise SN would not be violated. For $k > 2$, note that if any $P \cap R_i \cap R_j$ is empty, then $P \cap R_i$ and $P \cap R_j$ are disjoint, so that (a) holds; hence we can assume that every intersection of P and two R's is nonempty. If every pair of $P \cap R$'s intersects (i.e., (a) is not true), if some $P \cap R_i \cap R_j$ is a vertex it cannot be in $P \cap R_k$ for every $k \neq i, j$ (otherwise SN would not be violated); hence for some k, $P \cap R_i \cap R_j$, $P \cap R_j \cap R_k$ and $P \cap R_i \cap R_k$ must be disjoint, so that (b) holds. Similarly, if some $P \cap R_i \cap R_j$ is an edge, and some $P \cap R_k$ doesn't intersect it, then $P \cap R_i \cap R_j$, $P \cap R_j \cap R_k$, and $P \cap R_i \cap R_k$ must be disjoint, so that (b) holds. Finally, if all the $P \cap R$'s intersect the edge $P \cap R_i \cap R_j$, then there must exist R_k, R_l such that R_k meets that edge at one vertex, say p_k, and R_l meets it at the other vertex, say p_l (otherwise SN would not be violated). We now show that $P \cap R_k \cap R_l$ is either disjoint from $P \cap R_i$ or disjoint from $P \cap R_j$. If not, suppose $P \cap R_k \cap R_l$ meets $P \cap R_i$ at p_i and meets $P \cap R_j$ at p_j. p_i and p_j cannot be identical, because $P \cap R_k$ and $P \cap R_l$ meet $P \cap R_i \cap R_j$ at different vertices; thus $P \cap R_k \cap R_l$ must be the edge $p_i p_j$. For the same reason, p_i and p_j cannot be the same as p_k or p_l. Thus p_i, p_j, p_k, p_l are distinct vertices of P; moreover, $P \cap R_i \cap R_k$ contains p_i and p_k, so must be the edge $p_i p_k$; and $P \cap R_j \cap R_k$ contains p_j and p_k, so must be the edge $p_j p_k$. Thus $p_i p_j p_k$ is a triangle, and must be the face $P \cap R_k$. Similarly, $p_i p_j p_l$ is a triangle and is the face $P \cap R_l$; $p_i p_k p_l$ is a triangle and is the face $P \cap R_i$; and $p_j p_k p_l$ is a triangle and is the face $P \cap R_j$. Hence P is a tetrahedron; but this means that the intersection of R_i, R_j, R_k, R_l is empty, so that SN is not violated. Thus $P \cap R_k \cap R_l$ is either disjoint from $P \cap R_i$ or disjoint from $P \cap R_j$, say the former; but then $P \cap R_i \cap R_k$, $P \cap R_k \cap R_l$ and $P \cap R_i \cap R_l$ are disjoint, so that (b) holds.

If (a) is true (i.e. $P \cap R_i$ and $P \cap R_j$ are disjoint) then R_i, R_j is the minimal set of neighbors of P that violates SN. Let $\mathcal{P}' = \{P, R_i, R_j\}$. Obviously, $\mathcal{P}' \subseteq \mathcal{P}$ and $N_{\mathcal{P}'}(P) = P \cup R_i \cup R_j$. Let C be a closed curve in $N_{\mathcal{P}'}(P)$ that passes through each of the intersections $P \cap R_i$, $P \cap R_j$ and $R_i \cap R_j$. Suppose we can deform C so that it leaves any of the polyhedra, say R_i. Before C leaves R_i it had an arc from a point of $P \cap R_i$ to a point of $R_i \cap R_j$, passing through R_i. Hence just after C leaves R_i it must have points arbitrarily close to $P \cap R_i$ and $R_i \cap R_j$. Since $P \cap R_i$ is disjoint from R_j, the end of the arc that was previously in $P \cap R_i$; cannot be in R_j; hence it must be in P. Similarly, since $R_i \cap R_j$ is disjoint from P, the end that was previously close to $R_i \cap R_j$ cannot be in P; hence it must be in R_j. Since the arc no longer lies in R_i, to get from the endpoint in P to the endpoint in R_j it must pass through $P \cap R_j$. Just after the arc leaves R_i, it must be arbitrarily close to R_i; hence it cannot pass through $P \cap R_j$, which is disjoint from R_i. Thus the curve cannot leave R_i, and similarly it cannot leave R_j or P, so it cannot be reduced to a point, so that $N_{\mathcal{P}'}(P) = P \cup R_i \cup R_j$ is not simply connected.

If (a) is not true then (b) is true, so that there exist i, j, k such that $P \cap R_i \cap R_j$, $P \cap R_j \cap R_k$, and $P \cap R_k \cap R_i$ are disjoint. Let $\mathcal{P}' = \{P, R_i, R_j, R_k\}$. Since the R's violate SN, their intersection must be nonempty; in particular, $R_i \cap R_j \cap R_k$ is nonempty and so contains some vertex p. As noted earlier, since (a) does not hold, $P \cap R_i \cap R_j$, $P \cap R_j \cap R_k$, and $P \cap R_k \cap R_i$ must all be nonempty; let p_k p_i, p_j be vertices in these intersections, and let T be the tetrahedron defined by these four vertices. Note that p, p_i, p_j are all in R_k; p, p_j, p_k are all in R_i; p, p_k, p_i are all in R_j; and p_i, p_j, p_k are all in P. Thus each of P, R_i, R_j, R_k contains a face of T. On the other hand, since the R's violate SN, p is not in P, p_i is not in R_i, p_j is not in R_j, and p_k is not in R_k, so that none of P, R_i, R_j, R_k contains T. This also implies that T is nondegenerate. [Indeed, if its vertices were coplanar (or collinear), two of the triangles (possibly degenerate) defined by triples of the vertices would partially intersect; but this implies that the polyhedra containing these triples must partially intersect, contradicting normality, or that one of them contains the other and so contains all four of the vertices, contradiction.] Now the interior of T is surrounded by the faces of T; hence it is surrounded by $P \cup R_i \cup R_j \cup R_k (= N_{\mathcal{P}'}(P))$. If we can show that the interior of T is not contained in $P \cup R_i \cup R_j \cap R_k$, it will follow that $N_{\mathcal{P}'}(P)$ has a cavity, and hence is not simply connected.

As we just saw, no one of the polyhedra can contain (the interior of) T. If none of them intersects the interior, we are done. If one of them intersects the interior, since it cannot contain the entire interior, we are also done unless another one also intersects the interior. We shall show that the interior contains a vertex. Suppose R_i and R_j both intersect the interior. If $R_i \cap R_j$ is contained in the interior, the interior contains the vertices p and p_k. If $R_i \cap R_j$ intersects the interior but the interior contains no vertex, then $R_i \cap R_j$ intersects the surface of T. But $R_i \cap R_j \cap$ (the surface of T) is the line segment pp_k, which is an edge of T, so that no subset of that line segment intersects the interior of T, contradiction. We have thus proved that the interior of T contains a vertex, call it v. All four polyhedra cannot meet at v, since $\mathcal{P}'$ violates SN; and if only two of them meet at v, they cannot fill up the space around v; hence exactly three of them must meet at v, say R_i, R_j, and R_k. Let p_j be a vertex that is common to two of them (R_i and R_k). If vp_j is an edge, three polyhedra are needed to fill the space around it, and since p_j is not in R_j, the third polyhedron must be P, so that v is in all four polyhedra, contradiction. Hence $R_i \cap R_k$ must be a face F, and vp_j is a diagonal of F. Now F intersects the surface of T in the line segment pp_j. The sequence of edges of F from v to p_j, not passing through p, must all be in the interior of T; let vx be the first of these edges. If vx is common to R_i, R_j, and R_k, then the diagonal px of F is also common to R_i, R_j, and R_k, so that F is common to all three of them, which is impossible. But if vx is common to R_i, R_k, and P, then v is common to all four polyhedra, contradiction. □

4 Concluding remarks

We have defined intersection constraints on a set of convex polygons or polyhedra that must be satisfied to insure that the neighborhood of any polygon or polyhedron in the set is simply connected. It would be of interest to extend our results to more general classes of discrete spaces such as those studied in some of the references in [2].

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A set $\mathcal{P}$ of nondegenerate convex polygons P in R^2, or polyhedra P in R^3, will be called *normal* if the intersection of any two of the P's of $\mathcal{P}$ is a face (in the case of polyhedra), an edge, a vertex, or empty. $\mathcal{P}$ is called *strongly normal* (SN) if it is normal and, for all $P, P_1, \ldots, P_n$, if each P_i intersects P and $I = P_1 \cap \ldots \cap P_n$ is nonempty, then I intersects P. The union of the $P_i \in \mathcal{P}$ that intersect $P \in \mathcal{P}$ is called the *neighborhood* of P in $\mathcal{P}$, and is denoted by $N_{\mathcal{P}}(P)$. We prove that $\mathcal{P}$ is SN iff for any $\mathcal{P}' \subseteq \mathcal{P}$ and $P \in \mathcal{P}'$, $N_{\mathcal{P}'}(P)$ is simply connected. Thus SN characterizes sets $\mathcal{P}$ of polyhedra (or polygons) in which the neighborhood of any polyhedron, relative to any subset $\mathcal{P}'$ of $\mathcal{P}$, is simply connected. Tessellations of R^2 or R^3 into convex polygons or polyhedra are normal, but they may not be SN; for example, the square and hexagonal regular tessellations of R^2 are SN, but the triangular regular tessellation is not.

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